Assignment Problems with Weighted and Nonweighted Neighborhood Constraints in 3^6, 4^4 and 6^3 Tilings

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Abstract

We formulated a binary integer program to model the assignment problem stated as follows: the elements of given finite sets should be assigned to the compartments of a tiling (with finite number of compartments) such that the costs of having adjacent elements from different sets are minimized. We defined that two compartments are adjacent if and only if they share a common edge. In this paper, we considered the regular tilings of regular polygons in Euclidean plane.

An assignment problem can have weighted and nonweighted neighborhood constraints. Weights ω_g and $\omega_{\overline{g}}$ are assigned to sets g and $\overline{g}$, respectively. The cost of having an element from set g adjacent to an element of set $\overline{g}$ is computed as $|\omega_g - \omega_{\overline{g}}|$. In an assignment problem with weighted neighborhood constraint, the higher adjacency costs are minimized first.

In an assignment problem with nonweighted neighborhood constraint, the costs of the adjacencies between elements of different sets are simultaneously minimized, and the weights are considered as dummy only. The effect of the dummy weights can be removed by permuting the weights in the objective function of the binary integer program. The binary integer program associated with the assignment problem with nonweighted neighborhood constraint is computationally more expensive than the assignment problem with weighted constraint.

We can represent the tilings as graphs and the assignment problems as linearized binary integer programs. We presented illustrative examples showing the optimal solutions; however, optimal solutions may not be unique.

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Keywords: assignment problem, regular tilings, integer programming.

1 Introduction

In an assignment problem, elements of set A are assigned to the elements of set B subject to some criteria or constraints. The members of A can be persons, objects, machines or manufacturing plants; and B can contain tasks, locations or other objects. However, some constraints may exist because not all elements of A can be assigned to an element in B, or selected elements of A are best suited to be assigned to a specific element in B. This type of decision-making problem can be solved using mathematical programming, in particular, binary integer programming [7, 6]. Assignment problems can be seen in real-world scenarios such as assignment of workers to specific jobs, assignment of goods to storage areas in grocery stores, assignment of crops to certain areas in a plantation, assignment of patients with contagious diseases to hospital wards, or assignment of computers to connected network clusters.

In this study, we modeled the assignment problem of arranging elements that come from different finite sets into the compartments of a given tiling in Euclidean plane. We supposed that the tiling has a finite number of compartments. Each element is assigned to a compartment subject to the constraint that the sum of the costs of the adjacencies of the elements from different sets is minimized. The tilings that were considered are regular tilings of regular polygons, specifically 3^6 (triangular), 4^4 (square) and 6^3 (hexagonal) tilings [4, 3]. We defined that two elements are adjacent if they were assigned to adjacent compartments, and that two compartments are adjacent if and only if they share a common edge. For simplicity, we assumed that the total number of elements to be assigned is less than or equal to the number of compartments, and at most one element can be assigned to a compartment.

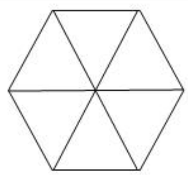
Figure 1. 3^6 tiling


Figure 2. 4^4 tiling

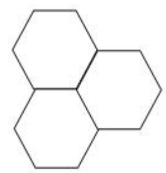
Figure 3. 6^3 tiling

An assignment problem can have weighted and nonweighted neighborhood constraints. Weights ω_g and $\omega_{\overline{g}}$ are assigned to sets g and $\overline{g}$, respectively. The cost of having an element from set g adjacent to an element of set $\overline{g}$ is computed as $|\omega_g - \omega_{\overline{g}}|$. In an assignment problem with weighted neighborhood constraint, we prioritize to minimize first the costs with higher value. In an assignment problem with nonweighted neighborhood constraint, the costs of the adjacencies between elements of different sets are simultaneously minimized without any priority, that is, we simply want that the elements from different sets are not adjacent.

Arranging the elements of sets in a tiling may be done manually using brute force (enumeration) if the number of elements and the number of compartments being considered are small. However, if the numbers of elements of the sets and compartments become large, then determining the optimal arrangement may be impractical for manual computation. The total number of possible arrangements (optimal and non-optimal) is equal to $\frac{\zeta!}{N_1!N_2!\ldots N_k!(\zeta - N_{total})!}$ where ζ is the number of compartments, N_g is the number of elements in set g for $g = 1, 2, \ldots, k$ and $N_{total} = N_1 + N_2 + \ldots + N_k$. Hence, we formulated binary integer programs that may help achieve the optimal solutions systematically.

1.1 Related Works

This study is related to location problems. Esteves et al. [2] were able to create a mathematical model that is expected to help solve the problem of overpopulation of bees by determining the optimal distribution of bee colonies in a specific location. They used mixed integer programming in formulating their model while they used graphs to visually represent the environment where the beehives will be distributed.

The assignment problem with nonweighted neighborhood constraint is related to graph coloring in which minimum number of colors are needed to paint a map with the restriction of having no same colors that are adjacent. In 2010, the paper of Diaby and Moustapha [1] used integer programming technique to solve graph coloring problems.

De Lara and Rabajante [5] presented a model for an assignment problem with weighted neighborhood constraint in 4^4 tilings. We extended their study to include nonweighted neighborhood constraint and to other tilings. In their paper, they came up with the following model (see Model Formulation section for the definition of parameters and variables):

Minimize

$$\sum_{i=1}^{r}\sum_{j=1}^{c-1}\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{gi(j+1)}\right|+\sum_{i=1}^{r-1}\sum_{j=1}^{c}\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{g(i+1)j}\right|$$

subject to

Constraint 1: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij}\begin{cases} = 0, & \text{if } ij\text{−th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

Constraint 2: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r}\sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 3: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

1.2 Linear Programming

A linear program is one of the optimization models which has linear objective and constraint expressions. This technique is of the form

$$\textit{Optimize } f(x_1, x_2, \ldots, x_n)$$

$$\textit{subject to} \quad \left.\begin{array}{c} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ g_n(x_1, x_2, \ldots, x_n) \end{array}\right\} \begin{array}{c} \leq \\ = \\ \geq \end{array} \left\{\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right.$$

Integer linear programs are linear programs in which the variables are integer-valued. Binary integer linear programs require that the variables be restricted to 0 or 1 only. Mixed integer linear programs may have integer and real-valued variables.

1.3 Conversion of NonLinear Objective Function to a Linear Program

If the objective function of a mathematical program is nonlinear such as absolute value function, then, if applicable, we should linearized it so as not to violate the general assumptions of linear programming.

For example, the objective function

$$\textit{Minimize } |f(x_1, x_2, \ldots, x_n)|$$

can be transformed into

$$\textit{Minimize } \alpha$$

$$\textit{subject to} \quad \begin{array}{l} f(x_1, x_2, \ldots, x_n) - \alpha \leq 0 \\ -f(x_1, x_2, \ldots, x_n) - \alpha \leq 0 \\ \alpha \in \mathbb{R}^{\oplus}. \end{array}$$

2 Model Formulation

2.1 Parameters and Variables

- Let the binary-valued decision variables be

$$x_{gij} = \begin{cases} 0, & \text{if an element from set } g \text{ is not assigned to the} \\ & \text{compartment at the } i\text{−th row and } j\text{−th column} \\ 1, & \text{otherwise} \end{cases}$$

  for $i = 1, 2, \ldots, r$ where r is the number of rows, and $j = 1, 2, \ldots, c$ where c is the number of columns. The decision variable x_{gij} is like an "on/off" variable — it will be "on" (that is equal to 1) if an element of set g is assigned to the ij-th compartment, and it will be "off" (that is equal to 0) if no element from set g is assigned to the ij-th compartment.

- Let N_g be the number of elements in set g for $g = 1, 2, \ldots, k$ where k is the number of sets. Let ω_g be the weight given to set g.

- Suppose $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$ be the dummy weights associated to the decision variables $y_1, y_2, \ldots, y_k, z_1, z_2, \ldots, z_k$, respectively, then define the relation $\rho_{(O\star)}$ as

$$\begin{aligned}\rho_{(O\star)}(|(a_1y_1 + a_2y_2 + \cdots + a_ky_k) - (b_1z_1 + b_2z_2 + \cdots + b_kz_k)|) \\ = \left|\kappa_{1(O\star)}\right| + \left|\kappa_{2(O\star)}\right| + \cdots + \left|\kappa_{k(O\star)}\right|\end{aligned}$$

  where

$$\begin{aligned}\kappa_{1(O\star)} &= (a_1y_1 + a_2y_2 + \cdots + a_{k-1}y_{k-1} + a_ky_k) \\ &\quad -(b_1z_1 + b_2z_2 + \cdots + b_{k-1}z_{k-1} + b_kz_k) \\ \kappa_{2(O\star)} &= (a_2y_1 + a_3y_2 + \cdots + a_ky_{k-1} + a_1y_k) \\ &\quad -(b_2z_1 + b_3z_2 + \cdots + b_kz_{k-1} + b_1z_k) \\ \kappa_{3(O\star)} &= (a_3y_1 + a_4y_2 + \cdots + a_1y_{k-1} + a_2y_k) \\ &\quad -(b_3z_1 + b_4z_2 + \cdots + b_1z_{k-1} + b_2z_k) \\ &\vdots \\ \kappa_{(k-1)(O\star)} &= (a_{k-1}y_1 + a_ky_2 + \cdots + a_{k-3}y_{k-1} + a_{k-2}y_k)\end{aligned}$$

$$
\begin{aligned}
&-(b_{k-1}z_1 + b_k z_2 + \cdots + b_{k-3}z_{k-1} + b_{k-2}z_k) \\
\kappa_{k(O\star)} &= (a_k y_1 + a_1 y_2 + \cdots + a_{k-2}y_{k-1} + a_{k-1}y_k) \\
&-(b_k z_1 + b_1 z_2 + \cdots + b_{k-2}z_{k-1} + b_{k-1}z_k).
\end{aligned}
$$

The relation $\rho_{(O\star)}$ denotes the circular shift permutation of the dummy weights in objective function term $O\star$. This relation was used in modeling an assignment problem with nonweighted neighborhood constraint.

2.2 Assignment Problem with Weighted Neighborhood Constraint

For an assignment problem with weighted neighborhood constraint, we prioritize to minimize first the adjacency costs with higher value. For example, if we have three sets, say set 1 with weight equal to 1, set 2 with weight equal to 2 and set 3 with weight equal to 3, then we prioritize to minimize first the adjacency between set 1 and set 3 (with cost equal to 2) before minimizing the adjacencies between set 1 and set 2 (with cost equal to 1) and between set 2 and set 3 (with cost equal to 1).

We can introduce dummy compartments so that each row has equal number of compartments and each column also has equal number of compartments, see Figures 4 and 5 for illustration. We used this approach to simplify our summation notations in the binary integer programs.

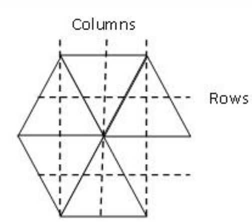

Figure 4. Triangular tiling with a missing compartment

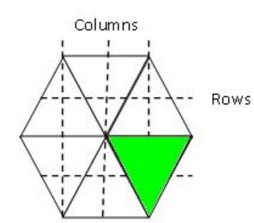

Figure 5. Triangular tiling with a dummy compartment

2.2.1 In 3^6 Tiling (Model 1)

For the 3^6 tiling, two types of tilings were considered — one that starts with adjacent columnar compartments (Model 1) and the other starts with non-adjacent columnar compartments (Model 2). See Figures 6 and 7 for illustration.

The tilings can be transformed into graphs in which the nodes represent the compartments and the edges represent the adjacencies among compartments, see Figure 8 for illustration. Now, the following is the formulated nonlinear binary integer model for the assignment problem with weighted neighborhood constraint in 3^6 tiling starting with adjacent columnar compartments:

Minimize

$$\sum_{i=1}^{r}\sum_{j=1}^{c-1}\left|\sum_{g=1}^{k}\omega_g x_{gij} - \sum_{g=1}^{k}\omega_g x_{gi(j+1)}\right| \quad (O1)$$

$$+\sum_{i=1}^{\lceil r/2\rceil-1}\sum_{j=1}^{\lceil c/2\rceil}\left|\sum_{g=1}^{k}\omega_g x_{g(2i)(2j)} - \sum_{g=1}^{k}\omega_g x_{g(2i+1)(2j)}\right| \quad (O2)$$

$$+\sum_{i=1}^{\lfloor r/2\rfloor}\sum_{j=1}^{\lfloor c/2\rfloor}\left|\sum_{g=1}^{k}\omega_g x_{g(2i-1)(2j-1)} - \sum_{g=1}^{k}\omega_g x_{g(2i)(2j-1)}\right| \quad (O3)$$

subject to

Constraint 1: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij} \begin{cases} = 0, & \text{if } ij\text{−th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

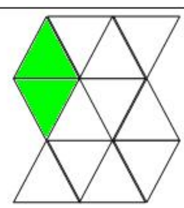

Figure 6. Triangular tiling starting with adjacent columnar compartments

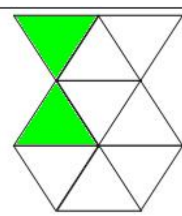

Figure 7. Triangular tiling starting with non-adjacent columnar compartments

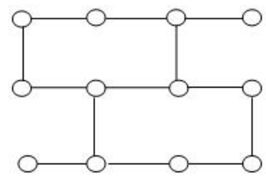

Figure 8. Graph representing the triangular tiling starting with adjacent columnar compartments

Constraint 2: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r} \sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 3: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function represents the costs of the row adjacencies. The term $O2$ represents the costs of the adjacencies in even columns, while the term $O3$ represents the costs of the adjacencies in odd columns. See Figure 9 for illustration.

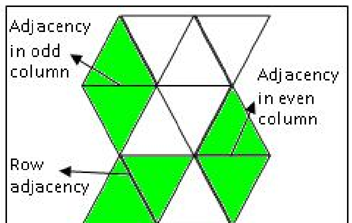

Figure 9. Row and Column Adjacencies

The first constraint guarantees that at most one element will be assigned to a compartment and that dummy compartments will not have any assigned element. The second constraint guarantees that each element will be assigned to a compartment, while the third constraint assures that the decision variable x_{gij} is binary-valued.

Notice that the objective function is composed of expressions in absolute value. We can use Integer Linear Programming techniques to solve the nonlinear model by linearizing the objective function. The following is the linearized binary integer program:

Minimize

$$\sum_{i=1}^{r}\sum_{j=1}^{c-1}\alpha_{ij} + \sum_{i=1}^{\lceil r/2\rceil-1}\sum_{j=1}^{\lceil c/2\rceil}\beta_{(2i)(2j)} + \sum_{i=1}^{\lfloor r/2\rfloor}\sum_{j=1}^{\lfloor c/2\rfloor}\gamma_{(2i-1)(2j-1)}$$

subject to

Constraint 1: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c-1$,

$$\sum_{g=1}^{k}\omega_g x_{gij} - \sum_{g=1}^{k}\omega_g x_{gi(j+1)} - \alpha_{ij} \leq 0$$

Constraint 2: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c-1$,

$$-\sum_{g=1}^{k}\omega_g x_{gij} + \sum_{g=1}^{k}\omega_g x_{gi(j+1)} - \alpha_{ij} \leq 0$$

Constraint 3: For $i = 1, 2, \ldots, \lceil r/2\rceil - 1$ and $j = 1, 2, \ldots, \lceil c/2\rceil$,

$$\sum_{g=1}^{k}\omega_g x_{g(2i)(2j)} - \sum_{g=1}^{k}\omega_g x_{g(2i+1)(2j)} - \beta_{(2i)(2j)} \leq 0$$

Constraint 4: For $i = 1, 2, \ldots, \lceil r/2\rceil - 1$ and $j = 1, 2, \ldots, \lceil c/2\rceil$,

$$-\sum_{g=1}^{k}\omega_g x_{g(2i)(2j)} + \sum_{g=1}^{k}\omega_g x_{g(2i+1)(2j)} - \beta_{(2i)(2j)} \leq 0$$

Constraint 5: For $i = 1, 2, \ldots, \lfloor r/2\rfloor$ and $j = 1, 2, \ldots, \lfloor c/2\rfloor$,

$$\sum_{g=1}^{k}\omega_g x_{g(2i-1)(2j-1)} - \sum_{g=1}^{k}\omega_g x_{g(2i)(2j-1)} - \gamma_{(2i-1)(2j-1)} \leq 0$$

Constraint 6: For $i = 1, 2, \ldots, \lfloor r/2 \rfloor$ and $j = 1, 2, \ldots, \lfloor c/2 \rfloor$,

$$-\sum_{g=1}^{k} \omega_g x_{g(2i-1)(2j-1)} + \sum_{g=1}^{k} \omega_g x_{g(2i)(2j-1)} - \gamma_{(2i-1)(2j-1)} \leq 0$$

Constraint 7: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij} \begin{cases} = 0, & \text{if } ij\text{−th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

Constraint 8: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r} \sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 9: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function of the nonlinear binary integer program was transformed to constraints 1 and 2 in the linearized program. The term $O2$ was transformed to constraints 3 and 4, and the term $O3$ to constraints 5 and 6. Constraints 1, 2, …, 9 has $r(c-1)$, $r(c-1)$, $(\lceil r/2 \rceil - 1) \lceil c/2 \rceil$, $(\lceil r/2 \rceil - 1) \lceil c/2 \rceil$, $\lfloor r/2 \rfloor \lfloor c/2 \rfloor$, $\lfloor r/2 \rfloor \lfloor c/2 \rfloor$, rc, k and rck subconstraints, respectively.

2.2.2 In 3^6 Tiling (Model 2)

The following are the sample graph illustration (Figure 10) and the formulated nonlinear binary integer model for the assignment problem with weighted neighborhood constraint in 3^6 tiling starting with non-adjacent columnar compartments:

Minimize

$$\sum_{i=1}^{r} \sum_{j=1}^{c-1} \left| \sum_{g=1}^{k} \omega_g x_{gij} - \sum_{g=1}^{k} \omega_g x_{gi(j+1)} \right| \quad (O1)$$

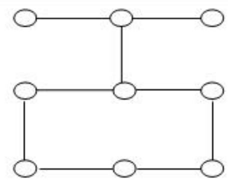

Figure 10. Graph representation of the triangular tiling starting with non-adjacent columnar compartments

$$+\sum_{i=1}^{\lceil r/2\rceil-1}\sum_{j=1}^{\lceil c/2\rceil}\left|\sum_{g=1}^{k}\omega_g x_{g(2i)(2j-1)}-\sum_{g=1}^{k}\omega_g x_{g(2i+1)(2j-1)}\right| \quad (O2)$$

$$+\sum_{i=1}^{\lfloor r/2\rfloor}\sum_{j=1}^{\lfloor c/2\rfloor}\left|\sum_{g=1}^{k}\omega_g x_{g(2i-1)(2j)}-\sum_{g=1}^{k}\omega_g x_{g(2i)(2j)}\right| \quad (O3)$$

subject to

Constraint 1: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij} \begin{cases} = 0, & \text{if } ij\text{−th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

Constraint 2: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r}\sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 3: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function represents the costs of the row adjacencies. The term $O2$ represents the costs of the adjacencies in odd columns, while the term $O3$ represents the costs of the adjacencies in even columns. We do the same process as in 3^6 Tiling (Model 1) to linearize the objective function.

2.2.3 In 6^3 Tiling

We define three types of adjacencies in a 6^3 tiling — row adjacency, column adjacency and diagonal adjacency, see Figure 11 for illustration. The following are the sample graph illustration (Figure 12) and the formulated nonlinear binary integer model for the assignment problem with weighted neighborhood constraint in 6^3 tiling:

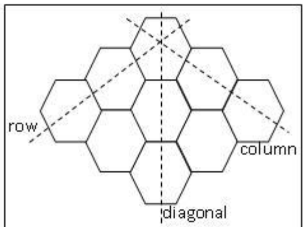

Figure 11. Hexagonal tiling and types of adjacencies

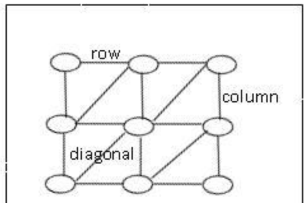

Figure 12.Graph representation of Hexagonal tiling

Minimize

$$\sum_{i=1}^{r}\sum_{j=1}^{c-1}\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{gi(j+1)}\right| \quad (O1)$$

$$+\sum_{i=1}^{r-1}\sum_{j=1}^{c}\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{g(i+1)j}\right| \quad (O2)$$

$$+\sum_{i=1}^{r-1}\sum_{j=2}^{c}\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{g(i+1)(j-1)}\right| \quad (O3)$$

subject to

Constraint 1: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij} \begin{cases} = 0, & \text{if } ij\text{−th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

Constraint 2: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r} \sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 3: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function represents the costs of the row adjacencies. The term $O2$ represents the costs of the column adjacencies, while the term $O3$ represents the costs of the diagonal adjacencies. We do the same process as in 3^6 Tiling (Model 1) to linearize the objective function.

2.3 Assignment Problem with Nonweighted Neighborhood Constraint

For an assignment problem with nonweighted neighborhood constraint, we simultaneously minimize the costs of the adjacencies between elements from different sets without considering any priority. That is, we only want to have the least number of adjacencies between elements from different sets without considering any weights. However, we cannot just remove the weights or let the weights assigned to all sets be equal, that is $|\omega_g - \omega_{\overline{g}}| = 0$ for any set g and $\overline{g}$, because this implies that we do not have any cost to minimize. To remedy this, we let the assigned weights ω_g for any set g to be considered as dummy, but we will later eliminate the effect of the dummy weights by using the relation $\rho_{(O\star)}$. For simplicity, we let $\omega_g = g$.

For example, refer to Figure 13, say we have sets $s1$, $s2$ and $s3$. In the first permutation, the adjacencies between elements of $s1$ and $s3$ should be minimized first because they have the largest adjacency cost ($cost = 2$). In the second permutation, the adjacencies between elements of $s1$ and $s2$ should be minimized first because they have the largest adjacency cost ($cost = 2$). In the third permutation, the adjacencies between elements of $s2$ and $s3$ should be minimized first because they have the largest adjacency cost ($cost = 2$). By doing this circular shift permutation, we eliminate the effect of the dummy weights. However, this strategy would make the linearized binary integer model associated to an assignment problem with nonweighted neighborhood constraint to have more subconstraints and to be computationally expensive than the model associated to an assignment problem with weighted neighborhood constraint.

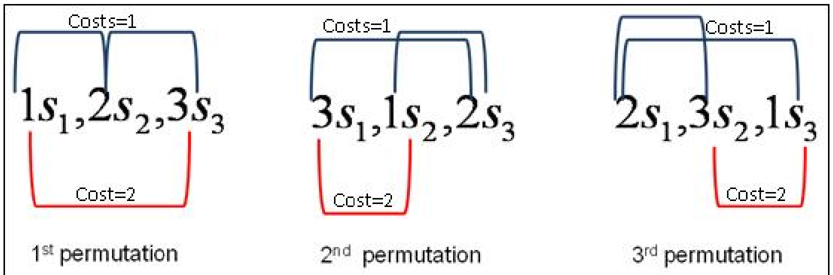

Figure 13: Circular shift permutation of the weights given three sets

2.3.1 In 3^6 Tiling (Model 1)

The following is the formulated nonlinear binary integer model for the assignment problem with nonweighted neighborhood constraint in 3^6 tiling starting with adjacent columnar compartments:

Minimize

$$\sum_{i=1}^{r}\sum_{j=1}^{c-1}\rho_{(O1)}\left(\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{gi(j+1)}\right|\right) \quad (O1)$$

$$+\sum_{i=1}^{\lceil r/2\rceil-1}\sum_{j=1}^{\lceil c/2\rceil}\rho_{(O2)}\left(\left|\sum_{g=1}^{k}\omega_g x_{g(2i)(2j)}-\sum_{g=1}^{k}\omega_g x_{g(2i+1)(2j)}\right|\right) \quad (O2)$$

$$+\sum_{i=1}^{\lfloor r/2\rfloor}\sum_{j=1}^{\lfloor c/2\rfloor}\rho_{(O3)}\left(\left|\sum_{g=1}^{k}\omega_g x_{g(2i-1)(2j-1)}-\sum_{g=1}^{k}\omega_g x_{g(2i)(2j-1)}\right|\right) \quad (O3)$$

subject to

Constraint 1: For $i=1,2,\ldots,r$ and $j=1,2,\ldots,c$,

$$\sum_{g=1}^{k} x_{gij}\begin{cases} =0, & \text{if } ij\text{–th compartment is a dummy compartment}\\ \leq 1, & \text{otherwise}\end{cases}$$

Constraint 2: For $g=1,2,\ldots,k$,

$$\sum_{i=1}^{r}\sum_{j=1}^{c} x_{gij}=N_g$$

Constraint 3: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function represents the costs of the row adjacencies. The term $O2$ represents the costs of the adjacencies in even columns, while the term $O3$ represents the costs of the adjacencies in odd columns.

The objective function is composed of expressions in absolute value. Now, the following is the linearized binary integer program:

Minimize

$$\sum_{i=1}^{r} \sum_{j=1}^{c-1} \sum_{h=1}^{k} \alpha_{hij} + \sum_{i=1}^{\lceil r/2 \rceil - 1} \sum_{j=1}^{\lceil c/2 \rceil} \sum_{h=1}^{k} \beta_{h(2i)(2j)} + \sum_{i=1}^{\lfloor r/2 \rfloor} \sum_{j=1}^{\lfloor c/2 \rfloor} \sum_{h=1}^{k} \gamma_{h(2i-1)(2j-1)}$$

subject to

Constraint 1: For $h = 1, 2, \ldots, k$, $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c-1$,

$$\kappa_{h(O1)} - \alpha_{hij} \leq 0$$

Constraint 2: For $h = 1, 2, \ldots, k$, $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c-1$,

$$-\kappa_{h(O1)} - \alpha_{hij} \leq 0$$

Constraint 3: For $h = 1, 2, \ldots, k$, $i = 1, 2, \ldots, \lceil r/2 \rceil - 1$ and $j = 1, 2, \ldots, \lceil c/2 \rceil$,

$$\kappa_{h(O2)} - \beta_{h(2i)(2j)} \leq 0$$

Constraint 4: For $h = 1, 2, \ldots, k$, $i = 1, 2, \ldots, \lceil r/2 \rceil - 1$ and $j = 1, 2, \ldots, \lceil c/2 \rceil$,

$$-\kappa_{h(O2)} - \beta_{h(2i)(2j)} \leq 0$$

Constraint 5: For $h = 1, 2, \ldots, k$, $i = 1, 2, \ldots, \lfloor r/2 \rfloor$ and $j = 1, 2, \ldots, \lfloor c/2 \rfloor$,

$$\kappa_{h(O3)} - \gamma_{h(2i-1)(2j-1)} \leq 0$$

Constraint 6: For $h = 1, 2, \ldots, k$, $i = 1, 2, \ldots, \lfloor r/2 \rfloor$ and $j = 1, 2, \ldots, \lfloor c/2 \rfloor$,

$$-\kappa_{h(O3)} - \gamma_{h(2i-1)(2j-1)} \leq 0$$

Constraint 7: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij} \begin{cases} = 0, & \text{if } ij\text{−th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

Constraint 8: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r} \sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 9: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function of the nonlinear binary integer program was transformed to constraints 1 and 2 in the linearized program. The term $O2$ was transformed to constraints 3 and 4, and the term $O3$ to constraints 5 and 6. Constraints 1, 2, …, 9 has $kr(c-1)$, $kr(c-1)$, $k(\lceil r/2 \rceil - 1) \lceil c/2 \rceil$, $k(\lceil r/2 \rceil - 1) \lceil c/2 \rceil$, $k \lfloor r/2 \rfloor \lfloor c/2 \rfloor$, $k \lfloor r/2 \rfloor \lfloor c/2 \rfloor$, rc, k and rck subconstraints, respectively.

2.3.2 In 3^6 Tiling (Model 2)

The following is the formulated nonlinear binary integer model for the assignment problem with nonweighted neighborhood constraint in 3^6 tiling starting with non-adjacent columnar compartments:

Minimize

$$\sum_{i=1}^{r} \sum_{j=1}^{c-1} \rho_{(O1)} \left(\left| \sum_{g=1}^{k} \omega_g x_{gij} - \sum_{g=1}^{k} \omega_g x_{gi(j+1)} \right| \right) \quad (O1)$$

$$+ \sum_{i=1}^{\lceil r/2 \rceil - 1} \sum_{j=1}^{\lceil c/2 \rceil} \rho_{(O2)} \left(\left| \sum_{g=1}^{k} \omega_g x_{g(2i)(2j-1)} - \sum_{g=1}^{k} \omega_g x_{g(2i+1)(2j-1)} \right| \right) \quad (O2)$$

$$+ \sum_{i=1}^{\lfloor r/2 \rfloor} \sum_{j=1}^{\lfloor c/2 \rfloor} \rho_{(O3)} \left(\left| \sum_{g=1}^{k} \omega_g x_{g(2i-1)(2j)} - \sum_{g=1}^{k} \omega_g x_{g(2i)(2j)} \right| \right) \quad (O3)$$

subject to

Constraint 1: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij} \begin{cases} = 0, & \text{if } ij-\text{th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

Constraint 2: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r} \sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 3: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function represents the costs of the row adjacencies. The term $O2$ represents the costs of the adjacencies in odd columns, while the term $O3$ represents the costs of the adjacencies in even columns. We do the same process as in 3^6 Tiling (Model 1) to linearize the objective function.

2.3.3 In 4^4 Tiling

The following is the formulated nonlinear binary integer model for the assignment problem with nonweighted neighborhood constraint in 4^4 tiling:

Minimize

$$\sum_{i=1}^{r} \sum_{j=1}^{c-1} \rho_{(O1)} \left(\left| \sum_{g=1}^{k} \omega_g x_{gij} - \sum_{g=1}^{k} \omega_g x_{gi(j+1)} \right| \right) \quad (O1)$$

$$+ \sum_{i=1}^{r-1} \sum_{j=1}^{c} \rho_{(O2)} \left(\left| \sum_{g=1}^{k} \omega_g x_{gij} - \sum_{g=1}^{k} \omega_g x_{g(i+1)j} \right| \right) \quad (O2)$$

subject to

Constraint 1: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij} \begin{cases} = 0, & \text{if } ij\text{−th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

Constraint 2: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r} \sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 3: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function represents the costs of the row adjacencies, while the term $O2$ represents the costs of the column adjacencies, see Figure 14 for illustration. We do the same process as in 3^6 Tiling (Model 1) to linearize the objective function.

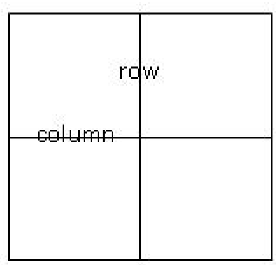

Figure 14. Row and column adjacencies in a square tiling

2.3.4 In 6^3 Tiling

The following is the formulated nonlinear binary integer model for the assignment problem with nonweighted neighborhood constraint in 6^3 tiling:

Minimize

$$\sum_{i=1}^{r}\sum_{j=1}^{c-1}\rho_{(O1)}\left(\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{gi(j+1)}\right|\right) \quad (O1)$$

$$+\sum_{i=1}^{r-1}\sum_{j=1}^{c}\rho_{(O2)}\left(\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{g(i+1)j}\right|\right) \quad (O2)$$

$$+\sum_{i=1}^{r-1}\sum_{j=2}^{c}\rho_{(O3)}\left(\left|\sum_{g=1}^{k}\omega_g x_{gij}-\sum_{g=1}^{k}\omega_g x_{g(i+1)(j-1)}\right|\right) \quad (O3)$$

subject to

Constraint 1: For $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$,

$$\sum_{g=1}^{k} x_{gij} \begin{cases} = 0, & \text{if } ij\text{−th compartment is a dummy compartment} \\ \leq 1, & \text{otherwise} \end{cases}$$

Constraint 2: For $g = 1, 2, \ldots, k$,

$$\sum_{i=1}^{r}\sum_{j=1}^{c} x_{gij} = N_g$$

Constraint 3: For $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, c$ and $g = 1, 2, \ldots, k$,

$$x_{gij} \in \{0, 1\}$$

The term $O1$ in the objective function represents the costs of the row adjacencies. The term $O2$ represents the costs of the column adjacencies, while the term $O3$ represents the costs of the diagonal adjacencies. We do the same process as in 3^6 Tiling (Model 1) to linearize the objective function.

3 Illustrative Examples

Let us suppose there are three groups and denote G1, G2 and G3 to be elements from Group 1, Group 2 and Group 3, respectively. The distribution of elements for each group is shown in Table 1.

Table 1: Distribution of elements per group.

Group	Number of elements
Group 1	3
Group 2	4
Group 3	5

Now we want to assign the members of the given groups to the compartments of a tiling such that the costs of having adjacent elements from different sets are minimized. Assume $\omega_g = g$.

The tilings to be considered for assignment problem with weighted neighborhood constraint are shown in Figures 15 and 16. The tilings to be considered for assignment problem with nonweighted neighborhood constraint are shown in Figures 17 and 18.

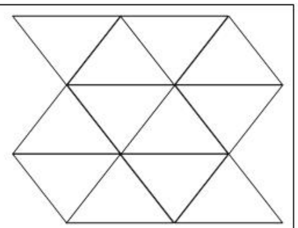

Figure 15. Design of the triangular tiling starting with non-adjacent columnar compartments

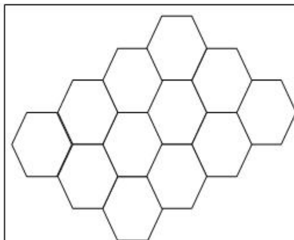

Figure 16. Design of the hexagonal tiling

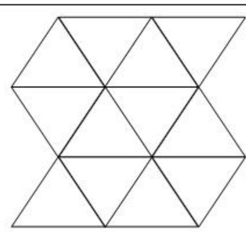

Figure 17. Design of the triangular tiling starting with adjacent columnar compartments

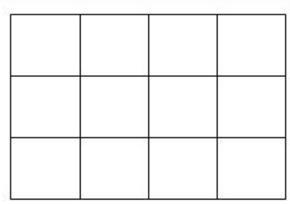

Figure 18. Design of the square tiling

The optimal solutions (not necessarily unique) to our illustrative examples are shown in Figures 19-22. The optimal solutions were determined using General Algebraic Modeling System (GAMS) version 23.7.

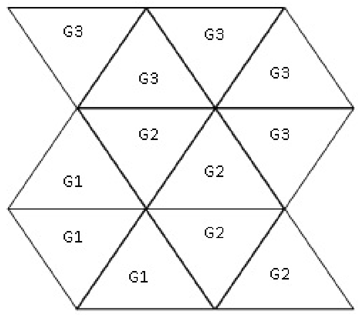

Figure 19. Optimal solution, triangular tiling starting with non-adjacent columnar compartments

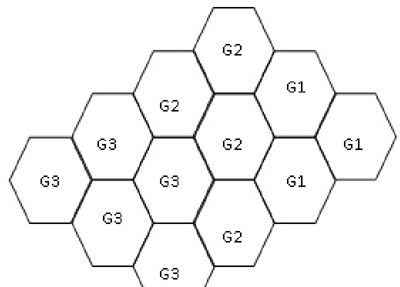

Figure 20. Optimal solution, hexagonal tiling

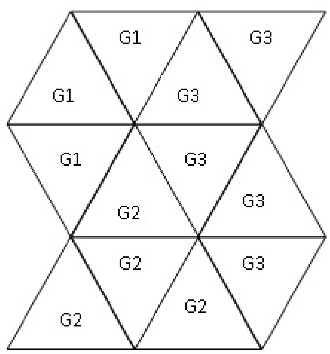

Figure 21. Optimal solution, triangular tiling starting with adjacent columnar compartments

Figure 22. Optimal solution, square tiling

4 Summary

In this study, binary integer models were formulated for the assignment problem stated as: given a finite number of k sets and finite number of M compartments of a regular polygonal tiling, each N_g element in set g $(g = 1, 2, \ldots, k)$ should be arranged in the polygonal tiling such that $N_1 + N_2 + \cdots + N_k \leq M$ and the costs of having adjacent elements from different sets are minimized. Two neigborhood constraints were considered — weighted and nonweighted. We used the idea of circular shift permutation to model the assignment problems with nonweighted neighborhood constraint. The formulated binary integer programs were linearized, i.e. transformed to linear programming models, since the objective functions have absolute value expressions. This study can be extended to include other kinds of adjacencies and other types of tilings.

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